Chapter 3 Mass properties

Bachelor Program in AUTOMATION ENGINEERING
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3.1 Centre of mass

- Consider a system composed of N particles belonging to the same plane and rigidly connected;
- Assuming a reference (Cartesian coordinate system)frame (x,y)attached to this system;
- Gravity center or center of mass G of a body:

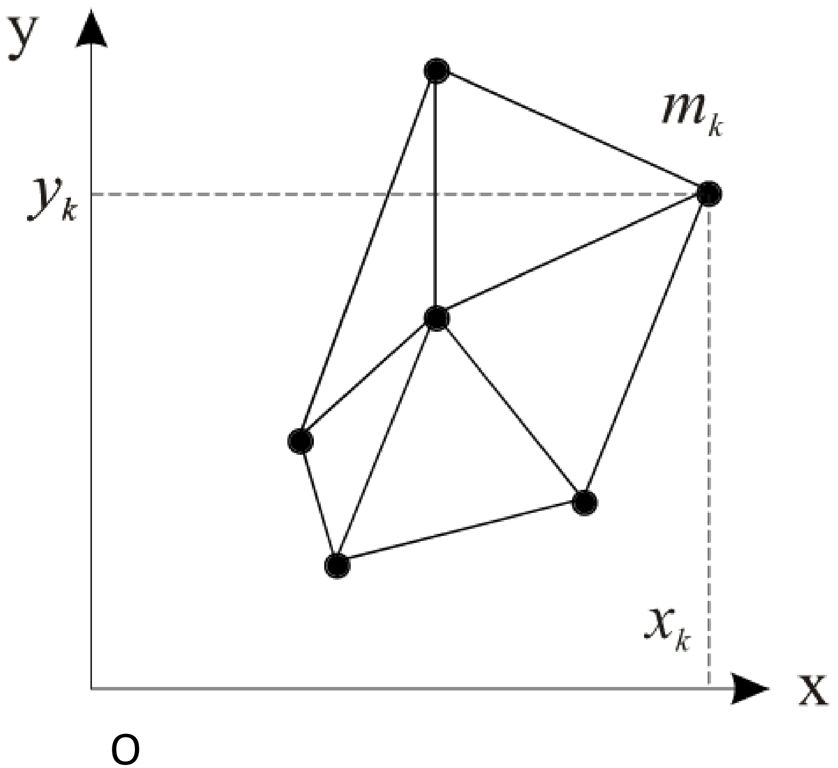


Fig. 3.1

$$M = \sum_{1}^{N} m_k ; \qquad x_G = \sum_{1}^{N} \frac{m_k x_k}{M}; \qquad y_G = \sum_{1}^{N} \frac{m_k y_k}{M}; \qquad (3.1)$$

Gravity center of a continuous rigid body

- The mass of an infinitesimal element= infinitesimal element of volume dV* the material density ρ, then dm= ρ dV

$$M = \int_V \rho(x,y,z)\,dV; \quad x_G = \frac{1}{M}\int_V \rho(x,y,z)\,x\,dV; \quad y_G = \frac{1}{M}\int_V \rho(x,y,z)\,y\,dV \qquad (3.2)$$

an homogeneous body (i.e. ρ =constant) and with a constant thickness h,

the infinitesimal mass becomes dm=ρhdA

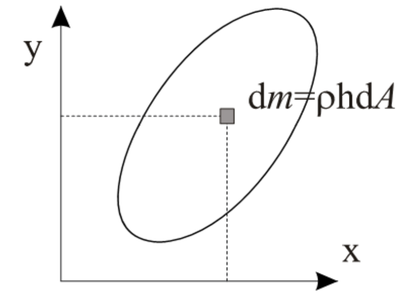


Fig. 3.2

$$M = \rho h \int_A dA; \quad x_G = \frac{\rho h}{M}\int_V x\,dA; \quad y_G = \frac{\rho h}{M}\int_V y\,dA$$

Symmetric body(geometry center)

- When the homogeneous body(constant thickness) is symmetric, the center of mass lies on the symmetry axis.

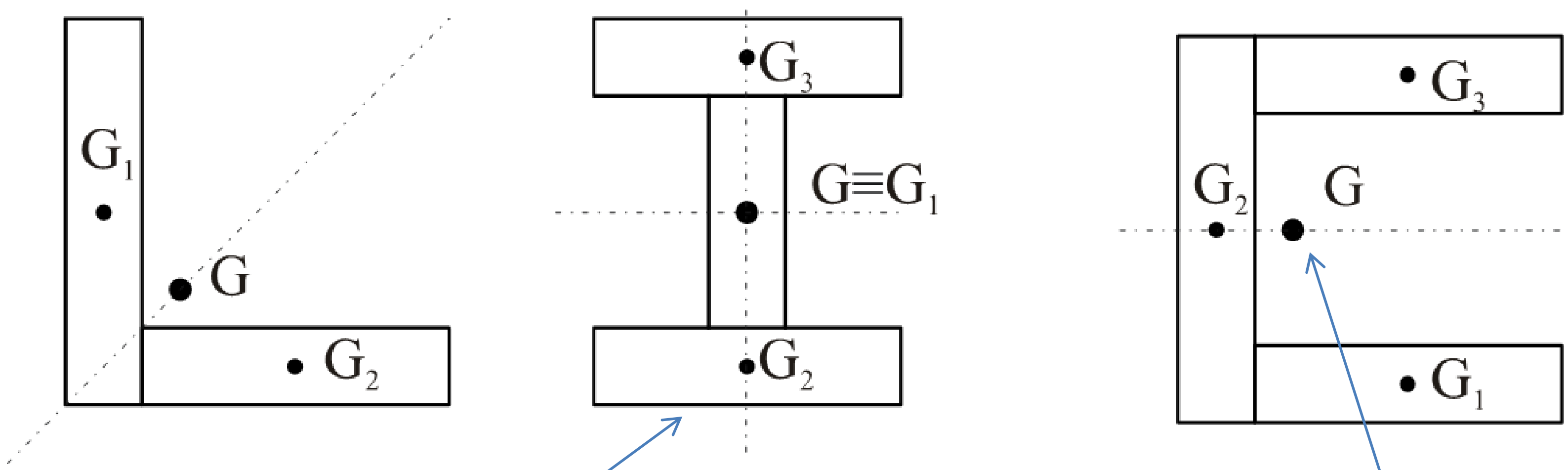


Fig. 3.3

double symmetry, the center of mass is at the axes intersection

Decomposed as known center of mass, then the whole gravity center:

$$x_G = \sum_{i=1}^{N} \frac{m_k x_{G_k}}{M}; \qquad y_G = \sum_{i=1}^{N} \frac{m_k y_{G_k}}{M}$$

3.2 The center of mass as center of gravity (c.g.)

- the center of mass is also the point of application of the force due to gravity;
- Consider simplicity, assume that a rigid N particles body on a plane;
- the modulus of the moment of the gravity forces of the composing particles:

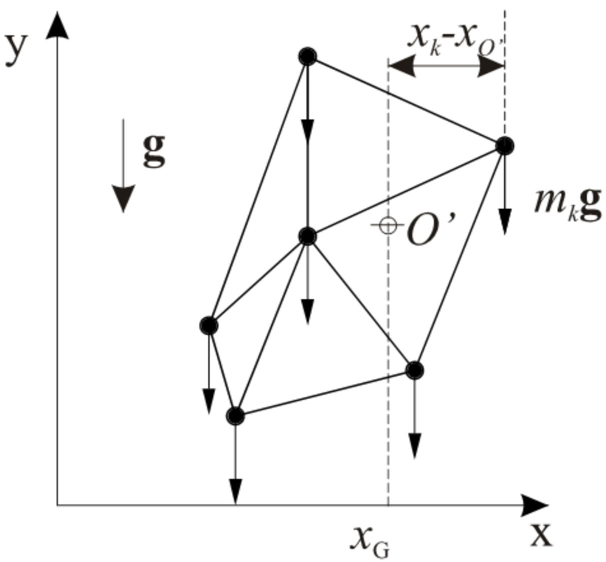


Fig. 3.4

Let g be the acceleration of gravity , parallel to the y axis (fig. 3.4)

$$M_{O'} = \sum_{k=1}^{N} (x_k - x_{O'}) m_k g$$

To move O', until horizontal balance

By equating $M_{O'}$ to zero we have

$$\sum_{k=1}^{N} x_k m_k = x_{O'} \sum_{k=1}^{N} m_k$$

Weight force passes mass center

$$\sum_{k=1}^{N} x_k m_k = x_{O'} \sum_{k=1}^{N} m_k$$

$$\sum_{k=1}^{N} m_k = M$$

$$x_{O'} = \frac{\sum_{k=1}^{N} x_k m_k}{M} = x_G$$

- x value of balancing point O’

Similar procedure: to rotate the rigid body and coordinate system anti-clockwise

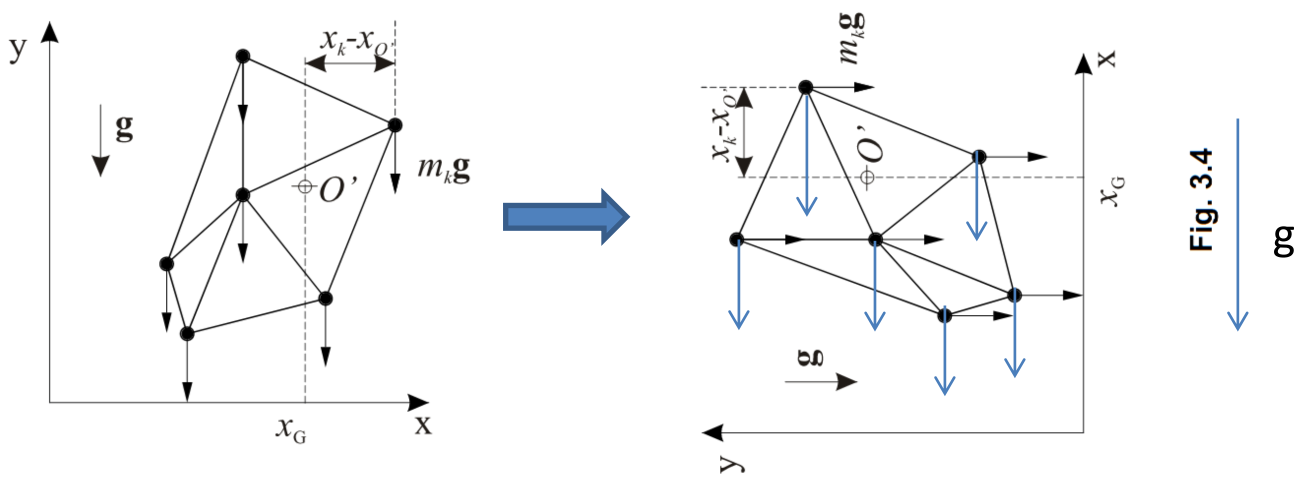


Fig. 3.4

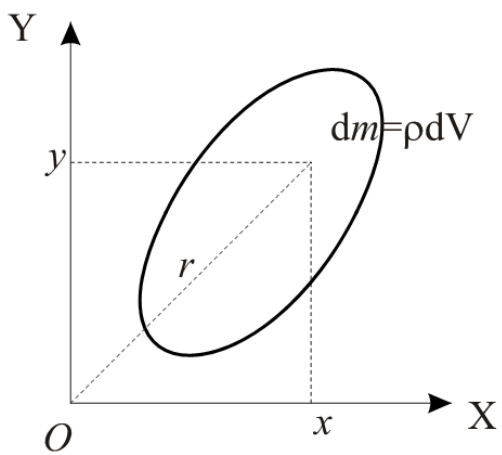
3.3 Mass moment of inertia

- The mass moment of inertia of a rigid body is indicative of the body mass distribution;

around an axis normal to the plane OXY and passing through O is defined

$$J_O = \int_V r^2 dm = \int_V \left(x^2 + y^2\right)\rho dV \quad (3.8)$$

homogeneous body of constant thickness h

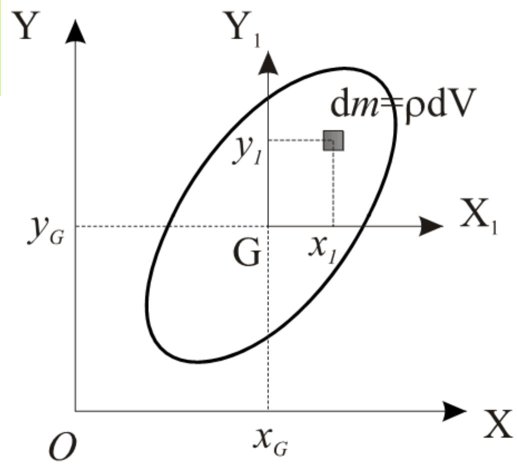


$$J_O = \rho h \int_A r^2 dA = \rho h \int_A \left(x^2 + y^2\right) dA \quad (3.9)$$

Mass Center reference

$$J_O = \rho h \int_A r^2 dA = \rho h \int_A \left(x^2 + y^2\right) dA \quad (3.9)$$

- using a reference system GX1Y1 having its origin in the center of mass G



$$J_O = \int_V \left(\left(x_G + x_1\right)^2 + \left(y_G + y_1\right)^2 \right) \rho dV$$

developing and reordering terms,

$$J_O = \left(x_G^2 + y_G^2\right) \int_V \rho dV + 2x_G \int_V x_1 \rho dV + 2y_G \int_V y_1 \rho dV + \int_V \left(x_1^2 + y_1^2\right) \rho dV$$

Body mass M

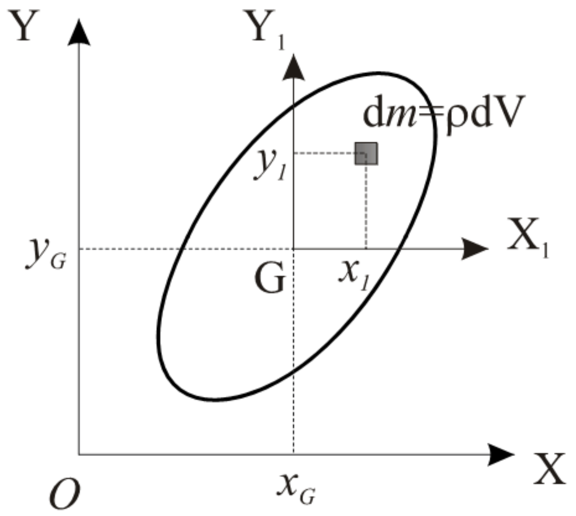
the whole integrate of local mass center G is 0 ⇒ $\int_V x_1 \rho dV = \int_V y_1 \rho dV = 0$

integral $\int_V \left(x_1^2 + y_1^2\right) \rho dV$ $= J_G$ represents the mass moment of inertia J_G with respect to an axis through G

Further analysis

- To study the rigid body dynamics it is convenient to assume the axis passing through the mass center G as a reference axis to calculate the mass moment of inertia.

$$J_O = \left(x_G^2 + y_G^2\right)M + J_G \qquad (3.12)$$



if a body can be decomposed in N simple components of mass M_k, of known centers of mass G_k and moments of inertia J_{Gk}

$$J_G = \sum_{k=1}^{N}\left(J_{G_k} + M_K \overline{GG_k}^2\right) \qquad (3.13)$$

the distance of the center of mass of the k-th simple body from the overall center G.

Radius of gyration

- The moment of inertia around the center G is also assigned by means of the mass M of he body and a geometrical quantity called radius of gyration ρ

$$J_G = M\rho^2 \qquad (3.14)$$

Physical meanings are coherent

$$J_O = \int_V r^2 dm = \int_V \left(x^2 + y^2\right)\rho dV \qquad (3.8)$$

Application 1:moment of inertia of a homogeneous annulus

- Given an annulus of constant thickness h and density ρ, with internal and external radius R2 and R1;
- the moment of inertia around an axis normal to the plane of the ring and passing through G is

$$J_G = \int_0^{2\pi} \int_{R_2}^{R_1} r^2 \rho h dr\, r d\theta = \rho h \int_0^{2\pi} d\theta \int_{R_2}^{R_1} r^3 dr = 2\pi \rho h \left(\frac{R_1^4}{4} - \frac{R_2^4}{4} \right)$$

$$= \frac{\pi h \left(R_1^2 - R_2^2 \right)}{2} \left(R_1^2 + R_2^2 \right)$$

$$= M \frac{\left(R_1^2 + R_2^2 \right)}{2}$$

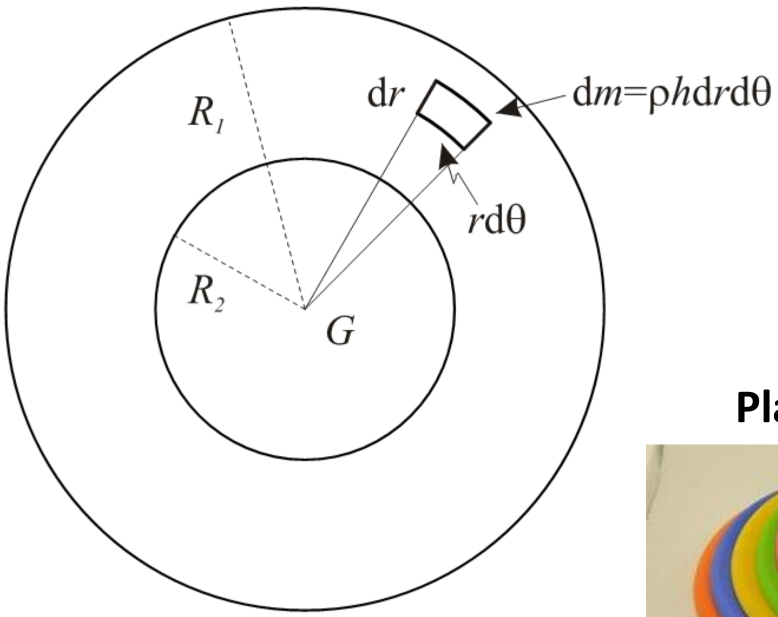


Fig. 3.6

Plastic ring Frisbee

Discussion

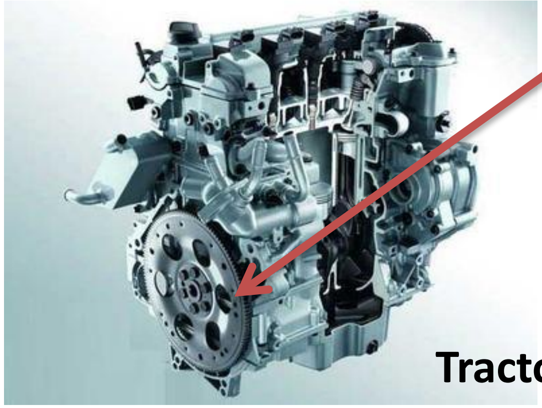
Considering the case of a disc with R_2=0 and R_1=R we have

$$J_G = M\frac{R^2}{2}; \quad \rho = \frac{R}{\sqrt{2}}$$

In the case of R_2=R_1=R, i.e. negligible radial thickness, we obtain

$$J_G = MR^2 \qquad \rho = R$$

- to maximize the moment of inertia, a body similar to a thin ring should be used



Tractor engine

Flywheel : mechanical components frequently used in various types of machinery assembled with crank shaft

tractor

Application 2: moment of inertia of a homogeneous bar

- consider a homogeneous bar of length L and mass M (fig. 3.7).
- By using a reference system placed at the geometrical center of the bar, coincident with the center of mass due to symmetry, and by indicating with m =M/L, its mass per unit length

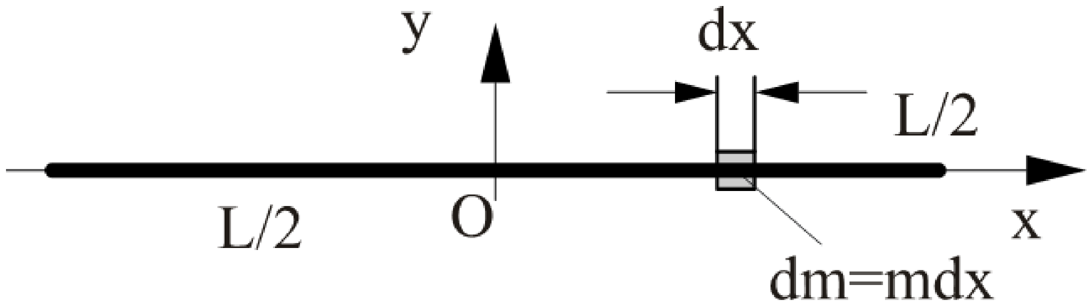


Fig. 3.7

According to definition of mass moment of inertia

$$J_O = \int_V r^2 dm \qquad (3.8)$$

$$J_G = \int_{-L/2}^{L/2} x^2 m dx = m\left[\frac{x^3}{3}\right]_{-L/2}^{L/2} = \frac{mL^3}{12} = \frac{ML^2}{12} \qquad (3.15)$$