Chapter 2 Kinematics of particles and rigid bodies
(2-4) Kinematics of particles: Cartesian approach using relative motions

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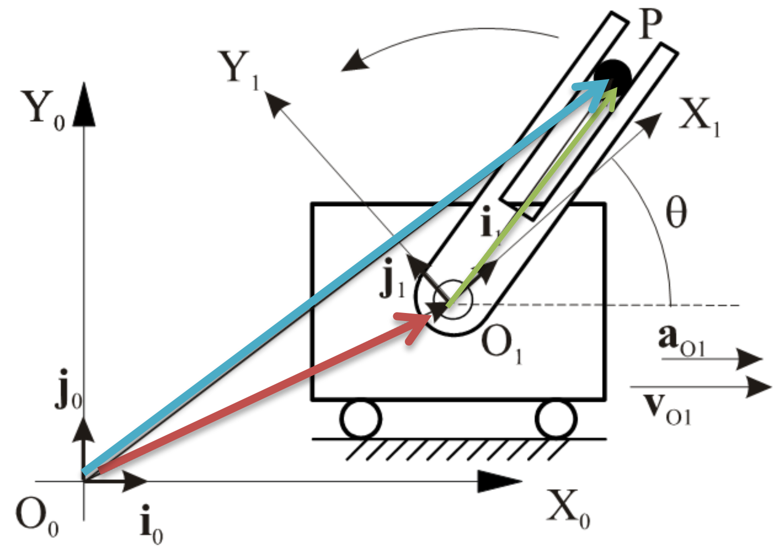
2.3 Kinematics of a particle: Cartesian approach using relative motions

Analyze in two different reference systems

Point P is moving with respect to the system $O_1X_1Y_1$

Also rotating and translating in a fixed (absolute) reference frame $O_0X_0Y_0$.



P in the absolute reference system

$$\mathbf{P} = (P - O) = (P - O_1) + (O_1 - O)$$

$$= (\mathbf{i}_1 x_1 + \mathbf{j}_1 y_1) + (\mathbf{i}_0 x_0 + \mathbf{j}_0 y_0)$$

the time first derivative

$$\mathbf{v}_P = \frac{d(P-O)}{dt} = \mathbf{i}_0 \dot{x}_0 + \mathbf{j}_0 \dot{y}_0 + \frac{d\mathbf{i}_1}{dt} x_1 + \mathbf{i}_1 \frac{dx_1}{dt} + \frac{d\mathbf{j}_1}{dt} y_1 + \mathbf{j}_1 \frac{dy_1}{dt}$$

the absolute velocity of the origin O_1

the velocity of P with respect of the moving frame O1X1Y1

To simplify the equation with complex number system

- In complex expression, unit vector:

unit vector time derivatives

$$\dot{\theta} = \omega$$

$$\mathbf{i}_1 = e^{i\theta} \Rightarrow \frac{d\mathbf{i}_1}{dt} = i\dot{\theta}e^{i\theta} = \dot{\theta}e^{i(\theta+\frac{\pi}{2})} = \mathbf{j}_1\dot{\theta} \Rightarrow \frac{d\mathbf{i}_1}{dt} = \mathbf{j}_1\omega$$

$$\mathbf{j}_1 = e^{i(\theta+\frac{\pi}{2})} = ie^{i\theta} \Rightarrow \frac{d\mathbf{j}_1}{dt} = i^2\dot{\theta}e^{i\theta} = -\dot{\theta}e^{i\theta} = -\mathbf{i}_1\dot{\theta} \Rightarrow \frac{d\mathbf{j}_1}{dt} = -\mathbf{i}_1\omega$$

$$\mathbf{v}_P = \frac{d(P-O)}{dt} = \mathbf{i}_0\dot{x}_0 + \mathbf{j}_0\dot{y}_0 + \frac{d\mathbf{i}_1}{dt}x_1 + \mathbf{i}_1\frac{dx_1}{dt} + \frac{d\mathbf{j}_1}{dt}y_1 + \mathbf{j}_1\frac{dy_1}{dt}$$

$$\Rightarrow \mathbf{v}_P = \mathbf{i}_0\dot{x}_0 + \mathbf{j}_0\dot{y}_0 + \mathbf{i}_1\dot{x}_1 + \mathbf{j}_1\dot{y}_1 + \omega\left(-\mathbf{i}_1 y_1 + \mathbf{j}_1 x_1\right)$$

Analysis

- Another expression in cross product

$$\omega\left(-\mathbf{i}_1 y_1+\mathbf{j}_1 x_1\right)=\boldsymbol{\omega}\times\left(P-O_1\right) \Longrightarrow$$

$$\boldsymbol{\omega}=\mathbf{k}\omega$$

$\omega=\dot{\theta}$ and $\mathbf{k}$ is a unit vector orthogonal to the directory plane

$$\mathbf{k}\omega\times\left(P-O_1\right)=\begin{bmatrix}\mathbf{i}_1 & \mathbf{j}_1 & \mathbf{k}\\ 0 & 0 & \omega\\ x_1 & y_1 & 0\end{bmatrix}=\omega\left(\mathbf{j}_1 x_1-\mathbf{i}_1 y_1\right)$$

Analysis

- Velocity terms analysis

$$\mathbf{v}_P = \mathbf{i}_0\dot{x}_0 + \mathbf{j}_0\dot{y}_0 + \mathbf{i}_1\dot{x}_1 + \mathbf{j}_1\dot{y}_1 + \omega\left(-\mathbf{i}_1 y_1 + \mathbf{j}_1 x_1\right) \qquad (2.38)$$

contribution due to the motion of the origin O1 of the moving reference frame:

$$\mathbf{i}_0\dot{x}_0 + \mathbf{j}_0\dot{y}_0$$

contribution due to the rotation of the moving reference frame:

$$\omega\left(-\mathbf{i}_1 y_1 + \mathbf{j}_1 x_1\right)$$

relative velocity of P with respect to the moving reference frame:

$$\mathbf{i}_1\dot{x}_1 + \mathbf{j}_1\dot{y}_1$$

Velocity components

- Further , velocity components

$$\mathbf{i}_0\dot{x}_0 + \mathbf{j}_0\dot{y}_0 = \mathbf{v}_{O_1}$$

$$\omega\left(-\mathbf{i}_1 y_1 + \mathbf{j}_1 x_1\right) = \boldsymbol{\omega}\times\left(P - O_1\right)$$

$$\mathbf{i}_1\dot{x}_1 + \mathbf{j}_1\dot{y}_1 = \mathbf{v}_{PO_1}$$

To summarize the expression of the absolute velocity of a point P

$$\mathbf{v}_P = \mathbf{v}_{O_1} + \boldsymbol{\omega}\times\left(P - O_1\right) + \mathbf{v}_{PO_1}$$

Absolute acceleration of point P

- time derivative of eq. (2.38) gives the absolute acceleration of point P

$$\mathbf{v}_P = \mathbf{i}_0\dot{x}_0 + \mathbf{j}_0\dot{y}_0 + \mathbf{i}_1\dot{x}_1 + \mathbf{j}_1\dot{y}_1 + \omega\left(-\mathbf{i}_1 y_1 + \mathbf{j}_1 x_1\right) \tag{2.38}$$

$$\mathbf{a}_P = \frac{d\mathbf{v}_P}{dt} = \mathbf{i}_0\ddot{x}_0 + \mathbf{j}_0\ddot{y}_0 + \frac{d\mathbf{j}_1}{dt}\omega x_1 + $$

$$+\mathbf{j}_1\dot{\omega}x_1 + \mathbf{j}_1\omega\dot{x}_1 - \frac{d\mathbf{i}_1}{dt}\omega y_1 - \mathbf{i}_1\dot{\omega}y_1 - \mathbf{i}_1\omega\dot{y}_1 + \tag{2.40}$$

$$+\frac{d\mathbf{i}_1}{dt}\dot{x}_1 + \mathbf{i}_1\ddot{x}_1 + \frac{d\mathbf{j}_1}{dt}\dot{y}_1 + \mathbf{j}_1\ddot{y}_1$$

Analysis with acceleration terms

acceleration due to the translation motion of the moving reference frame

$$\mathbf{i}_0\ddot{x}_0+\mathbf{j}_0\ddot{y}_0=\mathbf{a}_{O_1}$$

acceleration component due to the rotation of the moving reference frame

$$\frac{d\mathbf{j}_1}{dt}\omega x_1-\frac{d\mathbf{i}_1}{dt}\omega y_1=-\omega^2\left(\mathbf{i}_1x_1+\mathbf{j}_1y_1\right)=-\omega^2\left(P-O_1\right)$$

acceleration component due to the rotation of the moving reference frame

$\mathbf{i}_1\omega\dot{y}_1$

$$\mathbf{j}_1\dot{\omega}x_1-\mathbf{i}_1\dot{\omega}y_1=\dot{\omega}\left(\mathbf{j}_1x_1-\mathbf{i}_1y_1\right)=\dot{\boldsymbol{\omega}}\times\left(P-O_1\right)$$

relative acceleration of P with respect to the moving reference frame

$$\mathbf{i}_1\ddot{x}_1+\mathbf{j}_1\ddot{y}_1=\mathbf{a}_{PO_1}$$

Coriolis component of acceleration

$$2\mathbf{j}_1\omega\dot{x}_1-2\mathbf{i}_1\omega\dot{y}_1=2\boldsymbol{\omega}\times\mathbf{v}_{PO_1}$$

$$\left.\begin{array}{l}\frac{d\mathbf{i}_1}{dt}\dot{x}_1\\ \frac{d\mathbf{i}_1}{dt}=\mathbf{j}_1\omega\end{array}\right\}\ \frac{d\mathbf{i}_1}{dt}\dot{x}_1=\mathbf{j}_1\omega\dot{x}_1$$

$$\left.\begin{array}{l}\frac{d\mathbf{j}_1}{dt}\dot{y}_1\\ \frac{d\mathbf{j}_1}{dt}=-\mathbf{i}_1\omega\end{array}\right\}\ \frac{d\mathbf{j}_1}{dt}\dot{y}_1=-\mathbf{i}_1\omega\dot{y}_1$$

Analysis with acceleration terms

the expression of the absolute acceleration of a point P, the Coriolis theorem

$$\mathbf{a}_P = \mathbf{a}_{O_1} - \omega^2 (P - O_1) + \dot{\boldsymbol{\omega}} \times (P - O_1) + \mathbf{a}_{PO_1} + 2\boldsymbol{\omega} \times \mathbf{v}_{PO_1}$$

If the moving reference frame had a translation motion,
eqs.(2.39) and (2.41) reduce to the simple forms

$$\mathbf{v}_P = \mathbf{v}_{O_1} + \boldsymbol{\omega} \times (P - O_1) + \mathbf{v}_{PO_1} \qquad (2.39)$$

$$\Downarrow$$

$$\mathbf{v}_P = \mathbf{v}_{O_1} + \mathbf{v}_{PO_1}$$

$$\mathbf{a}_P = \mathbf{a}_{O_1} - \omega^2 (P - O_1) + \dot{\boldsymbol{\omega}} \times (P - O_1) + \mathbf{a}_{PO_1} + 2\boldsymbol{\omega} \times \mathbf{v}_{PO_1} \qquad (2.41)$$

$$\Downarrow$$

$$\mathbf{a}_P = \mathbf{a}_{O_1} + \mathbf{a}_{PO_1}$$

2.3.2 Comparison with the complex numbers approach

- position of point P is given by the sum of two complex numbers

$$(P-O)=r_1e^{i\varphi}+r_2e^{i(\theta+\alpha)}$$

the position of the origin O1

the relative position of P on the moving plane

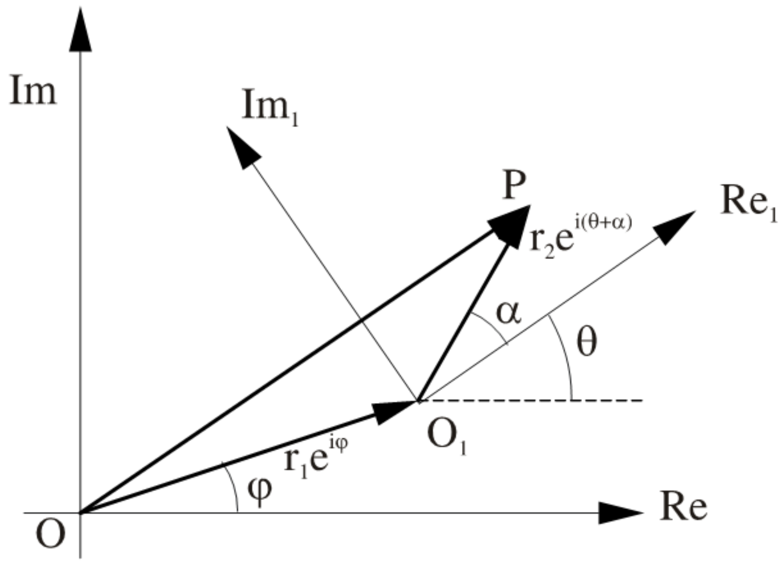


Fig. 2.23

time derivative

$$\overline{v}_P=\dot{r}_1e^{i\varphi}+i\dot{\varphi}r_1e^{i\varphi}+\dot{r}_2e^{i(\theta+\alpha)}+i\left(\dot{\theta}+\dot{\alpha}\right)r_2e^{i(\theta+\alpha)} \qquad (2.44)$$

Reordering terms

$$\overline{v}_P=\left(\dot{r}_1+ir_1\dot{\varphi}\right)e^{i\varphi}+ir_2\dot{\theta}e^{i(\theta+\alpha)}+\left(\dot{r}_2+ir_2\dot{\alpha}\right)e^{i(\theta+\alpha)} \qquad (2.45)$$

- To reorder (2.44), then

$$\overline{v}_P = \left(\dot{r}_1 + i r_1 \dot{\varphi}\right) e^{i\varphi} + i r_2 \dot{\theta} e^{i(\theta+\alpha)} + \left(\dot{r}_2 + i r_2 \dot{\alpha}\right) e^{i(\theta+\alpha)} \qquad (2.45)$$

- Comparing with (2.39), terms meanings:

$$\mathbf{v}_P = \mathbf{v}_{O_1} + \boldsymbol{\omega} \times (P - O_1) + \mathbf{v}_{PO_1} \qquad (2.39)$$

$$\left(\dot{r}_1 + i r_1 \dot{\varphi}\right) e^{i\varphi} \rightarrow \mathrm{v}_{O_1}$$

velocity due to the translation motion of the moving reference frame

$$i r_2 \dot{\theta} e^{i(\theta+\alpha)} \rightarrow \boldsymbol{\omega} \times \left(P - O_1\right)$$

velocity due to the rotation motion of the moving reference frame

$$\left(\dot{r}_2 + i r_2 \dot{\alpha}\right) e^{i(\theta+\alpha)} \rightarrow \mathrm{v}_{PO_1}$$

velocity of point P relative to the moving reference system

Special cases

- If the moving reference frame is only translating and if r_2 represents the distance between two points of the same rigid body, then : $\dot{\theta} = \dot{r}_2 = 0$

eq. (2.45) becomes:

$$\overline{v}_P = \left(\dot{r}_1 + i r_1 \dot{\varphi}\right) e^{i\varphi} + i r_2 \dot{\alpha} e^{i(\theta+\alpha)}$$

If the moving reference frame is only rotating around its origin O_1

$$\overline{v}_P = i r_2 \dot{\alpha} e^{i(\theta+\alpha)} + \left(\dot{r}_2 + i r_2 \dot{\theta}\right) e^{i(\theta+\alpha)}$$

Acceleration

- Time derivate

$$\bar{v}_P = \left(\dot{r}_1 + i r_1 \dot{\varphi}\right) e^{i\varphi} + i r_2 \dot{\theta} e^{i(\theta+\alpha)} + \left(\dot{r}_2 + i r_2 \dot{\alpha}\right) e^{i(\theta+\alpha)} \qquad (2.45)$$

$$\Downarrow$$

$$\begin{aligned}\bar{a}_P = & \left(\ddot{r}_1 + i\dot{r}_1\dot{\varphi} + i r_1 \ddot{\varphi}\right) e^{i\varphi} + i\dot{\varphi}\left(\dot{r}_1 + i r_1 \dot{\varphi}\right) e^{i\varphi} + \\ & + \left(i\dot{r}_2\dot{\theta} + i r_2 \ddot{\theta}\right) e^{i(\theta+\alpha)} - \left(\dot{\theta} + \dot{\alpha}\right) r_2 \dot{\theta} e^{i(\theta+\alpha)} + \\ & + \left(\ddot{r}_2 + i\dot{r}_2\dot{\alpha} + i r_2 \ddot{\alpha}\right) e^{i(\theta+\alpha)} + i\left(\dot{\theta} + \dot{\alpha}\right)\left(\dot{r}_2 + i r_2 \dot{\alpha}\right) e^{i(\theta+\alpha)}\end{aligned}$$

Comparisons to (2.41) allow us to find the following meanings

$$\mathbf{a}_P = \mathbf{a}_{O_1} - \omega^2 \left(P - O_1\right) + \dot{\boldsymbol{\omega}} \times \left(P - O_1\right) + \mathbf{a}_{PO_1} + 2\boldsymbol{\omega} \times \mathbf{v}_{PO_1} \qquad (2.41)$$

Acceleration term

- Term Meanings:

$$\left(\ddot{r}_1 + 2i\dot{r}_1\dot{\varphi} + ir_1\ddot{\varphi} - r_1\dot{\varphi}^2\right)e^{i\varphi} \quad \rightarrow \mathrm{a}_{O1}$$

$$-\dot{\theta}^2 r_2 \dot{\theta} e^{i(\theta+\alpha)} \quad \rightarrow -\omega^2\left(P - O_1\right)$$

$$ir_2\ddot{\theta}e^{i(\theta+\alpha)} \quad \rightarrow \dot{\omega} \times \left(P - O_1\right)$$

$$\left(\ddot{r}_2 + 2i\dot{r}_2\dot{\alpha} + ir_2\ddot{\alpha} - r_2\dot{\alpha}^2\right)e^{i(\theta+\alpha)} \quad \rightarrow \mathrm{a}_{PO_1}$$

$$2i\dot{\theta}\left(\dot{r}_2 + ir_2\dot{\alpha}\right)e^{i(\theta+\alpha)} \quad \rightarrow 2\omega \times \mathrm{v}_{PO_1}$$